Pion and η' strings

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In this paper we construct a string-like classical solution, the pion string, in the linear sigma model. We then study the stability of the pion string and find that it is unstable in the parameter space allowed experimentally. We also speculate on the existance of an unstable eta prime string, associated with the spontaneous breakdown of the anomalous $U_A(1)$ symmetry in QCD at high temperatures. The implications of the pion and eta prime strings for cosmology and heavy ion collisions are briefly mentioned. [S0556-2821(98)03414-6]

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Strings, as classical solutions in theories with spontaneously broken symmetries, play an important role in both particle physics and cosmology. Thus it is of crucial importance to know if strings can exist in realistic models of strong and electroweak interactions. Recently, in an inspiring paper, Vachaspati [1] showed that string-like structures exist in the standard model of electroweak theory. In this paper, we will consider strings in models with spontaneously broken chiral symmetry of QCD. First of all, we explicitly construct a classical solution, the pion string, in the linear sigma model. Similar to the Z string, the pion string is not topologically stable. Then we argue for the existence of an eta prime string at high temperatures.

In recent years, the $SU_L(2)\times SU_R(2)\sim O(4)$ linear sigma model has been often used as a model of hadron dynamics for chiral symmetry breaking in QCD, in particular in studies of the physics associated with the chiral phase transition [2] and of the disoriented chiral condensates [3] in heavy ion collisions. The Lagrangian density of this model is

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\sigma)^2+\frac{1}{2}(\partial_\mu\vec{\pi})^2-\frac{\lambda}{4}(\sigma^2+\vec{\pi}^2-f_\pi^2)^2+H\sigma. \quad (1)$$

In this paper, for simplicity, we consider the chiral limit $H=0$. In our discussion of the pion string it proves convenient to define the new fields

$$\phi=\frac{\sigma+i\pi^0}{\sqrt{2}}, \quad (2)$$

$$\pi^\pm=\frac{\pi^1\pm i\pi^2}{\sqrt{2}}. \quad (3)$$

The linear sigma model in Eq. (1) now can be rewritten as

$$\mathcal{L}=(\partial_\mu\phi)^*\partial^\mu\phi+\partial_\mu\pi^+\partial^\mu\pi^- -\lambda\left(\pi^+\pi^-+\phi^*\phi-\frac{f_\pi^2}{2}\right)^2. \quad (4)$$

During chiral symmetry breaking, the field ϕ takes on a nonvanishing vacuum expectation value, which breaks $SU_L(2)\times SU_R(2)$ down to $SU_{L+R}(2)$. This results in a massive sigma and three massless Goldstone bosons. In addition, we will demonstrate below that there is a static unstable string-like solution, the pion string.

For static configurations in Eq. (4), the energy functional is given by

$$E=\int d^3x\left[\vec{\nabla}\phi^*\vec{\nabla}\phi+\vec{\nabla}\pi^+\vec{\nabla}\pi^- +\lambda\left(\pi^+\pi^-+\phi^*\phi-\frac{f_\pi^2}{2}\right)^2\right]. \quad (5)$$

The time independent equations of motion are

$$\nabla^2\phi=2\lambda\left(\pi^+\pi^-+\phi^*\phi-\frac{f_\pi^2}{2}\right)\phi, \quad (6)$$

$$\nabla^2\pi^+=2\lambda\left(\pi^+\pi^-+\phi^*\phi-\frac{f_\pi^2}{2}\right)\pi^+. \quad (7)$$

The pion string solution extremizing the energy functional in Eq. (5) is given by

$$\phi=\frac{f_\pi}{\sqrt{2}}\rho(r)e^{in\theta}, \quad (8)$$

$$\pi^\pm=0, \quad (9)$$

where the coordinates r and θ are polar coordinates in the x-y plane, and the integer n is the winding number. In the following discusion, we will restrict ourselves to $n=1$.

Substituting Eq. (8) into Eq. (6), we obtain the equation of motion for $\rho(r)$:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\rho(r)-\frac{1}{r^2}\rho(r)=\lambda f_\pi^2(\rho^2-1)\rho(r). \quad (10)$$

The boundary conditions for $\rho(r)$ are

$$r\rightarrow 0, \quad \rho(r)\rightarrow 0, \quad (11)$$

$$r\rightarrow\infty, \quad \rho(r)\rightarrow 1. \quad (12)$$

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In principle, $\rho(r)$ can be determined numerically by solving Eq. (10) with boundary conditions of Eqs. (11) and (12). But it is simpler to adopt a variational approach. Following the method of Hill, Hodges and Turner [4], we make an ansatz of the following form:

$$\rho(r)=(1-e^{-\mu r}), \tag{13}$$

where μ is a variational parameter.

Adopting expression (13) as a variational ansatz, it follows that the energy per unit length of the string is

$$E_{(pion\ string)}=\frac{1}{4}\pi f_\pi^2+\pi f_\pi^2 I_\theta(\mu,R)+\frac{\pi}{2}\frac{89}{144}\lambda f_\pi^2\frac{f_\pi^2}{\mu^2}, \tag{14}$$

where $I_\theta(\mu,R)=\int_0^\infty (dr/r)(1-e^{-\mu r})^2\approx\ln(\mu R)$ is logarithmically divergent, as expected in a theory with global symmetry breaking. Thus, we introduce a large-scale cutoff R. However, the R dependence will disappear upon varying with respect to μ:

$$\left.\frac{\partial I_\theta(\mu,R)}{\partial\mu}\right|_{R\to\infty}=\frac{1}{\mu}. \tag{15}$$

The solution for μ is obtained by varying the energy functional of the pion string in Eq. (14) with respect to μ:

$$\mu^2=\lambda\frac{89}{144}f_\pi^2. \tag{16}$$

Substituting Eq. (16) into Eq. (14), the mass of the pion string per unit length is

$$E_{(pion\ string)}=\left[\frac{3}{4}+I_\theta(\mu,R)\right]\pi f_\pi^2\approx\left[\frac{3}{4}+\ln(\mu R)\right]\pi f_\pi^2. \tag{17}$$

The pion string is not topologically stable, since any field configuration can be continuously deformed to the vacuum. To study the stability of the pion string, we consider infinitesimal perturbations of the field $\pi^\pm$ and check if the variation in the energy is positive or negative.

Discarding terms of cubic and higher orders in $\pi^\pm$, we find

$$E=E_{(pion\ string)}+\delta E, \tag{18}$$

where

$$\delta E=\int d^3x[\vec{\nabla}\pi^+\vec{\nabla}\pi^-+\lambda f_\pi^2(\rho^2-1)\pi^+\pi^-]. \tag{19}$$

Following Ref. [5], we consider an expansion of the $\pi^\pm$ fields in Fourier modes:

$$\pi^+=\chi_m(r)e^{im\theta}. \tag{20}$$

Inserting the expressions for the mth mode of $\pi^\pm$ in Eq. (19) gives

$$\delta E=\int 2\pi r dr\left[\left(\frac{\partial\chi_m}{\partial r}\right)^2+\frac{m^2}{r^2}\chi_m^2+\lambda f_\pi^2(\rho^2-1)\chi_m^2\right], \tag{21}$$

where the first term (the kinetic energy part) and the second term are always positive, but the third term, the potential energy, is negative. Notice that the second term gives the smallest contribution to the positive energy in δE for $m=0$; so we will focus on $m=0$.

Defining $\xi=f_\pi r,\chi=f_\pi R$, and setting $m=0$, δE becomes

$$\delta E=2\pi f_\pi^2\int \xi d\xi\left[\left(\frac{\partial R}{\partial\xi}\right)^2+\lambda(\rho^2-1)R^2\right]. \tag{22}$$

After an algebraic computation we can rewrite δE in the form

$$\delta E=2\pi f_\pi^2\int \xi d\xi R\hat{O}R, \tag{23}$$

where

$$\hat{O}=-\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right)+\lambda(\rho^2-1). \tag{24}$$

The question of the stability of the pion string reduces to checking if the eigenvalues of the operator $\hat{O}$ in its spectrum are negative, subject to the eigenfunction R satisfying the boundary conditions $R(\xi\to 0)\to$ const, and $R(\xi\to\infty)\to 0$.

To simplify the analysis, we take a variational approach making use of an ansatz of the form [4]

$$R=\sigma_0 e^{-\kappa\xi}(1+\kappa\xi+\kappa_1\xi^2+\kappa_2\xi^3), \tag{25}$$

where $\sigma_0,\kappa,\kappa_1,\kappa_2$ are dimensionless variational parameters. This ansatz has the correct short-distance limit. Note that κ^{-1} represents the size of the $\pi^\pm$ condensate on the pion string. By inserting Eq. (25) into Eq. (23) it is obvious that the negative term wins out if $\lambda\geqslant 1$. This implies that the pion string is unstable in the parameter space allowed [2] experimentally ($\lambda\sim 10$–20). It can be shown by numerical analysis that δE is positive only for very small values of λ($\lambda\leqslant 10^{-8}$), and hence the pion string is only stable for these values.

In the early universe and in heavy-ion collisions, pion strings are expected to be produced and to subsequently decay. Their lifetime can be estimated by considering their interactions with the surrounding plasma [6]. Based on a naive dimensional analysis, their lifetime τ should be proportional to the inverse of the corresponding temperature. For a strongly interacting theory such as the linear sigma model studied here, we expect that $\tau=O(1)T^{-1}$, where T is the temperature at the time when the chiral symmetry of QCD is restored.

Before concluding, let us speculate about the existence of an unstable η' string. In QCD, in the limit of massless quarks, there is an additional $U_A(1)$ chiral symmetry. This chiral symmetry, when broken by the quark condensate, predicts the existence of a Goldstone boson. There is, however, no such light meson. This is resolved by the Adler-Bell-

Jackiw $U_A(1)$ anomaly together with the properties of the non-trival vacuum structure of non-Abelian gauge theory, in particular QCD. The $U_A(1)$ symmetry is badly broken by instanton effects at zero temperature.

As the density of matter and/or the temperature increases, it is expected that the instanton effects will rapidly disappear [2,7], and one may have an additional $U_A(1)$ symmetry [in addition to $SU_L(2)\times SU_R(2)$] at the transition temperature of the QCD chiral symmetry.[1] When the $U_A(1)$ symmetry is broken spontaneously by the quark condensate, an η' string results. If the $U_A(1)$ symmetry is exact above the QCD chiral phase transition, then, as long as the effects of instantons can be neglected, the η' string is topologically stable. The η' string can form during the chiral phase transition of QCD. In the setting of cosmology, it may exist during a specific epoch below the QCD chiral symmetry breaking temperature during the evolution of the universe. In the context of heavy-ion collisions, it may exist in the plasma created by the collision during a period shortly after the cooling of the interaction region below the symmetry breaking scale. The strings then become unstable as the temperature decreases and when the instanton effects become substantial.

[1] Recent lattice simulations indicate qualitatively, however, that the $U_A(1)$ symmetry may not be completely restored at the temperature of the QCD chiral phase transition, but only at a somewhat higher (possibly infinite) temperature [8] (for calculations in the context of instanton models, see [9] and [10]). Thus, the η' string is similar to the axion string.

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